

Allocation of CO₂ emissions in petroleum refineries to petroleum joint products: A linear programming model for practical application

Alireza Tehrani Nejad M. *

Institut Français du Pétrole (I.F.P.), 1 et 4, avenue de Bois-Préau, 92852, Rueil-Malmaison, France

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Abstract

The allocation of CO₂ emissions associated with petroleum refineries to petroleum joint products is a necessary step in Well-to-Tank analysis in order to evaluate the environmental impacts of individual automotive fuels. Oil refining is essentially a joint production system and due to the complex nature of the process involved, it is very difficult to establish any noncontroversial allocation pattern for oil products. Under certain conditions, however, refinery linear programming models can provide a non-arbitrary additive allocation schema based on the marginal contribution of each oil product to the total CO₂ emissions. But in general, these conditions are not satisfied.

In this paper, by extending the LP approach to the optimal Simplex tableau, we propose an original two-stage methodology based on the marginal contribution of oil products and the production elasticity of unit processes to provide an additive CO₂ allocation scheme. We show that this procedure emerges from the equilibrium behavior of the refinery and is consistent with microeconomic theory. A numerical example is provided.

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1. Introduction

The question about industrial energy that this paper addresses is the allocation of the refineries' CO₂ emissions to petroleum joint products. This allocation procedure constitutes a necessary step

* Tel.: +33 1 47 52 60 00.

E-mail addresses: alireza.tehrani@ifp.fr, artehrani2000@yahoo.fr.

in evaluating the environmental impacts of automotive fuels in Well-to-Tank (WTT) studies. In fact, WTT is the first part of Well-to-Wheel (WTW) studies and consists in assessing the energy consumption and the resultant CO₂ emissions along fuel chains from the extraction of feedstock until the delivery of fuels to the vehicle tanks.¹ Since the WTT differences among automotive fuels are due exclusively to the refinery component, especial care should be taken on the allocation principle used to assess the contribution of oil products to the refinery energy use and CO₂ emissions. Oil refining is a joint production system and due to the complex nature of the process involved and the vast number of joint product outputs that are strongly correlated, it is very difficult to establish any noncontroversial allocation scheme for oil products.

In practice, allocation rules used so far for the petroleum-based fuel are traditionally based on two fundamental approaches: physical measures (mass, volume, energy or exergy² contents, molecular mass or other relevant parameters) or market value (gross sale value) or expected economic gain of individual oil products from a given refinery. Both of these approaches inevitably involve the use of arbitrary allocation rules.

Furoholt (1995) and Wang et al. (2004) point out that these allocation rules should be applied at the sub-process level within a refinery and not at the aggregate process level (i.e., the refinery level). This would consist in partitioning the refinery into different process units and then allocating the energy consumption and the resultant emissions from each process unit to the products from these units according to mass, energy content or market value of final and intermediate petroleum products. They show that, contrary to the aggregate process level, tracking energy use and emissions by individual refining process helps reveal some additional energy and emissions associated with certain refinery products that are otherwise overlooked with the refinery-level allocation.

Although the “process-level-based method captures process-dependent characteristics of fuel production within a petroleum refinery” (Wang et al., 2004) it is still open to discussion on two points. First, despite the important effort of tracking the energy use and emissions by individual refining process, this approach still suffers from using arbitrary rules at its final step. In this regard, Azapagic and Clift (1999a,b) show how these arbitrary measures break down in joint production industrial systems when they do not reflect the underlying physical causality and lead to flawed results. For instance, in a Life Cycle Assessment study for the Statfjord production platform for production of regular gasoline, Furoholt (1995) reports that, based on the volume criterion, only 0.5% of the total energy use and the resultant emissions is allocated to gasoline, whereas it is 81% based on the energy criterion and 57% based on detailed partitioning.

Second, the sub-process allocation approach provides an incomplete picture of the whole system as it ignores the complex interactions, interdependencies and synergies which exist among the refinery oil products and process units. As a consequence, this approach systematically assigns more energy use and CO₂ emissions to the oil products that utilize more process units. An illustrative example of this issue is gasoline and diesel which constitute the two main automotive oil products in WTT and WTW analysis. Most of the existing WTT studies overestimate the environmental burdens (energy use and CO₂ emissions) of gasoline due to the higher number of gasoline processing units in European refineries (an exception is the WTT report of CONCAWE

¹ WTW and WTT studies are categorised in retrospective and prospective approaches. Prospective studies are based on marginal data and consider the effects of different decisions. On the other hand, retrospective studies are based on average data and study the environmental accounting issues. Throughout this paper, we only focus on the retrospective approach.

² The exergy content of a system indicates its distance from the thermodynamic equilibrium. The higher the exergy content, the farther from the thermodynamic equilibrium (definition from, <http://www.holon.se/folke>).

and EUCAR, 2006). To show that this conclusion could be wrong, let us consider a standard refining scheme where catalytic reforming units convert low-octane naphthas into high-octane gasoline blending components called reformats with hydrogen as a by product. In response to a continuously increasing demand for diesel (at the expense of gasoline), European refineries have expanded the diesel fraction from oil refining beyond its optimum balance with gasoline yield (Kavalov and Petevs, 2004). This imbalance should worsen due to the tightening of oil product sulphur specifications³ which would also cause the reduction of a refinery's overall output. These changes are associated with higher energy use and CO₂ emissions at the refinery level (see e.g., CONCAWE, 2000, 2005). For cost reasons, it usually happens that catalytic reforming units operate at full capacity not in order to meet the gasoline demand (which is decreasing in Europe) but to meet the increasing hydrogen requirement of the refinery. Therefore, the additional energy use and resultant CO₂ emissions of catalytic reforming units should no longer be assigned to the reforming gasoline output (as would be the case in most of the WTT approaches) but must be solely allocated among the products using the output of hydrodesulphurization units (especially diesel and other middle distillates).

This point has some pretty interesting implications for some reforms to the taxation of petroleum products in Europe: due to a lower taxation of diesel compared to gasoline (unequal taxation of the two automotive fuels results in diesel costing about one dollar less per gallon in most European countries), diesel cars dominate new vehicle registrations. This has led to an additional growing demand for diesel (by an average of 4% per year) and a further reduction in gasoline consumption (by an average of 2% per year, over the past five years); and, these changes in domestic demand have driven a significant increase in global trade flows for refined products and blend stocks (Houdek, 2005). Encouraging gasoline demand by changing the existing tax policy (in favour of gasoline) and improving the gasoline engine would probably correct the imbalance.⁴ The economic logic behind these policy reforms would be strengthened if we show that the European fuel market's evolution (in terms of both quantity and quality) has resulted in higher energy use, CO₂ emissions and infrastructure costs associated with diesel production as compared to gasoline.

More sophisticated proposals to energy consumption and emission allocation have been developed based on the concept of duality in linear programming (LP). In fact, LP is a powerful mathematical tool which is frequently used to represent the complex scenario of production in the refinery. The advantages of this method compared to the process-level-based approach could be summarized in two points. First, the LP model depicts causality between various inputs and outputs in the refinery and allocates the whole energy consumption and resultant CO₂ emissions accordingly without having to use any arbitrary measures (Azapagic and Clift, 1998). Second, the information created through the duality in LP takes into account the complex interdependency and synergy effects among the unit processes and petroleum products in the refinery system. These two "well-behaved" characteristics of the LP-based allocations are consistent with the ISO 14041 recommendations⁵ and therefore relevant for environmental accounting purposes. Under certain assumptions, these allocation coefficients can be also used for fully assigning the refinery's energy use and CO₂ emissions to the oil products. In this particular case, the marginal

³ Europe, like the United States, is implementing more stringent rules for sulphur: gasoline and "on-road" diesel sulphur content is reduced to 50 ppm in 2005 and will be reduced to 10 ppm by January 2009 (CONCAWE, 2005).

⁴ Note that, gasoline engine contains larger unexplored potential for further development than the diesel engine (Kavalov and Peteves, 2004).

⁵ see Appendix.

allocation coefficients can be viewed as the average CO₂ contribution of oil products and be used in WTT and WTW analysis. In the literature, this approach has been known as the marginal allocation methodology (Azapagic and Clift, 1994, 1995, 1998, 1999a,b; Babusiaux, 2003). The major disadvantages of LP models, however, is the difficult access to the technical data to researchers outside of refineries. Most of the time, such data are propriety and generally available only through refinery consultants and operators (Wang et al., 2004).

In general, in refinery LP models, process capacity, input availability and some types of calibrating and institutional constraints destroy the additivity property of the LP-based allocations. Due to the fact that the value of any basic variable (as well as the objective function value⁶) is fully distributed among the active constraints of the model, the petroleum product allocation coefficients along, might underestimate or overestimate the total volume of the refinery's CO₂ emissions. This handicap is a valid objection to LP as an allocation tool in WTT and WTW studies and might limit its use for problems in which the objective is to assign unambiguously the whole refinery's energy use and the resultant emissions among the oil products.

This methodological-oriented paper is aimed to provide an original two-stage procedure, based on LP, to fully allocate the refinery's CO₂ emissions among the refinery's petroleum products. We show that this non-arbitrary (re)allocation mechanism is additive, unique and defensible due to its micro-economic justification. The reminder of this paper is organized as follows. Section 2 briefly reviews the issue of allocating the refinery's CO₂ emissions via linear programming. Notations are introduced and the need for a reallocation procedure is discussed. Section 3 delineates the suggested (re)allocation methodology. A numerical example is provided to illustrate the procedure. Last section concludes.

2. Linear programming and the allocation of CO₂ emissions in the refinery

The use of LP in the refining industry spans a period of well over 50 years. The modelling of the overall refinery operation from the crude oil arrival to the distribution of oil products has quickly given rise to very complex linear programs. The blending of gasoline was among the first popular applications of LP in refineries (Charnes et al., 1952). Esso Standard Oil Company was probably the first to publish a book in 1955 entitled "Linear Programming — The Solution of Refinery Problems" (Symonds, 1955). Today, designing new units, fixing the operating conditions, making a choice of feedstocks, improving the operations planning, oil product costing, policy analysis and forecasting are among the routine utilization of LP in oil refineries.

2.1. General linear model of the refinery

A refinery LP model contains the end-to-end configuration of the refinery with a detailed representation of processing units, blending facilities, power and utilities. Such a model describes complex interactions between different process units and is based on physical relations between crude oils, intermediate products, finished products, energy consumption and pollutant emissions of the refinery. Therefore, a refinery LP model "lends itself naturally to solving allocation in multiple-function systems according to the procedure recommended by ISO 14041" (Azapagic and Clift, 1998).

In this paper, we develop a static LP model to study the CO₂ emissions allocation problem for a price-taking refinery operating in a cost-minimizing environment. We assume that the refiner's

⁶ see, Tehrani Nejad M. and Michelot (submitted for publication) for a different but symmetric re-allocation procedure applied to the objective function value for joint cost allocation purposes.

objective is to satisfy his petroleum production target (in terms of both level and specification), denoted by the m -vector b , at minimum cost subject to the prevailing technology, commodity prices and (fixed) input availabilities. The term c is the given n -vector of acquisition input cost and includes all costs that are absorbed in a direct costing system. It includes the cost of crude oils and feedstocks, the operating variable cost (i.e., cost of catalysts, solvents and chemicals), the exchange cost of finished products and also an emission-permit price (cost) for the refinery's carbon dioxide emissions for the future years. Suppose that the technology for transforming limiting inputs into oil product outputs is represented by the fixed coefficient matrix A of dimensions $m \times n$, and a_{ij} denotes the value in row i and column j . We assume that the refiner is operating in a deterministic environment and uses n divisible input activities (unit processes), represented by a $n \times 1$ non negative vector x . For notational simplicity, the capacity of all unit processes x is supposed to be limited to \bar{x} .

The pollution that we are dealing with is the refinery's carbon dioxide emissions that come mainly from the burning fuel. One emission balance equation is therefore needed to capture the whole CO_2 emissions of the refinery activity: the variable ε corresponds to the total carbon dioxide emissions generated from the burning of fuel gas (ethane and propane), liquefied fuel (e.g., vacuum residue) and the coke of the catalytic cracker in the refinery, each of which being assigned a specific CO_2 emission coefficient, $E=[E_1, E_2, \dots, E_n]$. We assume that the CO_2 contents of the refinery fuels are proportional to their quantities.

Besides the oil product demand and capacity constraints, the most common types of other constraints are the material balance and product quality constraints. The latter guarantees the expected quality and technical requirement of finished products in blending problems such as octane number (for gasoline) or sulphur content (for gasoline, diesel and heavy fuel oils) and, the former represents the fact that the sum total of quantities going into some unit process or blending pool equals the sum total coming out. Mathematically, both of these types of constraint can be formulated as $Zx=0$. Since the right-hand-side (RHS) of these constraints are zero, they do not affect the additivity property of LP-based allocations. Therefore, without loss of generality, we omit them from our basic model. A detailed description of the refinery constraints is beyond the scope of this paper and can be found in [Saint-Antonin \(1998\)](#).

Given these preliminaries, we may now state the one-period LP model of the refiner as

$$\begin{cases} \min & c^T x \\ \text{s.t.} & Ax \geq b \quad (\text{product demand constraints}) \\ & E^T x - \varepsilon = 0 \quad (\text{CO}_2 \text{ balance equation}) \\ & x \leq \bar{x} \quad (\text{capacity constraints}) \\ & x \geq 0, \varepsilon \geq 0. \end{cases} \quad (1)$$

The dual specification is

$$\begin{cases} \max & b^T y - \lambda^T \bar{x} \\ \text{s.t.} & A^T y + E\mu \leq c + \lambda \\ & y \geq 0, \mu \geq 0, \lambda \geq 0. \end{cases} \quad (2)$$

We assume that both problems (1) and (2) have a feasible solution implying that they have optimal solutions. Throughout this paper, we also assume that the primal and dual problems are not degenerate so that the optimal variables are unique.

The dual variables y associated with the oil product demand constraints Ax can be interpreted as their marginal production costs: they measure the variation of the cost function whenever the demand for one oil product is increased by one unit, while keeping demands for all other oil products constant. Similarly, the dual variables λ associated with the unit process capacity constraints $x \leq \bar{x}$ can be interpreted as their opportunity costs (or economic rents) at the optimum: they measure the variation of the cost function whenever the availability of one capacity constraint is increased, *ceteris paribus*. The dual variable μ is the shadow price associated with the CO₂ balance constraint. Notice that due to the KKT optimality conditions, μ is constrained to remain always non negative⁷. Now we can interpret the constraint of the dual problem (2) as follows: for achieving minimum total variable cost, the refiner must choose the combination of input activities x that equates the value marginal products of inputs, the left-hand side of (2), to their corresponding economic relevant costs $c + \lambda$.

2.2. LP-based CO₂ allocation methodology

By analogy to the cost allocation (Thomas, 1974), any theoretically justified or non-arbitrary allocation method should be additive, unambiguous and defensible. The additivity property requires that total refinery CO₂ emissions be equal to the sum of the parts, meaning that all of the refinery's carbon dioxide is allocated among the oil products, no more no less. The unambiguity condition requires the uniqueness of the allocated parts, and the defensibility criterion, which is the most important one, needs to provide conclusive proof for choosing a particular allocation method among all possible alternatives.

This section attempts to describe that the LP-based allocation coefficients satisfy the three desired characteristics of a non-arbitrary allocation method. Let us recall that one of the interesting features of LP is to fully allocate the optimal value of the basic variables among the primal constraints. Within the framework of model (1), this means that the input-related emission $\varepsilon = E^T x$ is transformed into oil product- and unit process-related emissions⁸ (see relation (6)). Through this optimal transformation, the whole CO₂ emissions are non-arbitrary shared between both oil products and scarce unit processes based on the well-known neoclassical concept of marginal contribution.

To show this interesting result, we suppose a general solution, where $M = \{1, 2, \dots, m\}$ and $S = \{1, 2, \dots, s\}$, with $s \leq n$, are the sets of active demand constraints and scarce unit processes at the optimum. In words, all final product demands are satisfied $Ax = b$, but only a part of the capacity constraints are active. Moreover, we denote B the $(k \times k)$ basic matrix and β the set of basic index. The primal feasible solutions associated with the problem (1) are then defined as

$$x_B = B^{-1} \begin{bmatrix} b \\ 0 \\ \bar{x} \end{bmatrix} \quad (3)$$

where x_B represents the basic variables and B^{-1} corresponds to the basis inverse matrix which is pre-multiplied by the right-hand-side (RHS) vector. Any extreme point of the refinery technological possibility set, including the optimal extreme point, can be defined by relation (3). Therefore the following discussion is obviously valid for any basic feasible solution.

⁷ Note that, the Lagrange multiplier μ is associated with an equality constraint and, in general, one cannot say anything a priori about the sign of it. But here, some derivations allow us to determine the sign of μ .

⁸ We borrow these terms from Azapagic and Clift (1998, 1999b, 2000).

Now let us focus on the CO₂ emission variable ε . At the optimum, ε is a basic variable (because always positive) so an element of x_B that can be expressed as

$$\varepsilon = E_B^T x_B = E_B^T B^{-1} \begin{bmatrix} b \\ 0 \\ \bar{x} \end{bmatrix} \tag{4}$$

where the vector E_B contains the emission coefficients as they appear in the column of the basic index β . We rewrite the relation (4) as

$$\varepsilon = e_\varepsilon^T B^{-1} \begin{bmatrix} b \\ 0 \\ \bar{x} \end{bmatrix} \tag{5}$$

where e_ε is the ε th unit vector ($e_{\varepsilon t} = 0$ for $t \neq \varepsilon$ and $e_{\varepsilon \varepsilon} = 1$) and $e_\varepsilon^T B^{-1}$ corresponds to the row of B^{-1} associated with the basic CO₂ variable ε . The row-vector $e_\varepsilon^T B^{-1}$ contains several blocs which are referred to the slack variables of model (1). Rewriting Eq. (5), we get:

$$\varepsilon = \overbrace{\sum_{i \in M} \alpha_i b_i}^{\text{product-related}} + \overbrace{\sum_{j \in S} \gamma_j \bar{x}_j}^{\text{product-related}} \tag{6}$$

where α_i and γ_j belong to the row vector which relates ε respectively to the demand and capacity slack variables in the final Simplex tableau. Eq. (6) allocates the total CO₂ emissions of the refinery to the oil products b and limited (fixed) unit processes through the marginal allocation coefficients, α_i ($i \in M$) and γ_j ($j \in S$). According to the optimal technology (the optimal basis B), α_i ($i \in M$) and γ_j ($j \in S$) can be positive, negative or zero. Under the non-degeneracy assumption, these allocation coefficients are uniquely determined and directly obtainable from the optimal Simplex tableau (see, Tehrani Nejad (submitted for publication) for a degenerate case).

The economic interpretation of relation (6) is: when the production functions $Ax=b$ are linear homogeneous and exhibit constant return to scale, the attribution of the carbon dioxide emissions to primal constraints (i.e., oil products and limiting unit processes) according to their marginal contributions (i.e., α_i and γ_j) is exactly equal to the whole CO₂ emissions generated within the refinery. The partitioned emissions reflect the underlying technical interdependencies embodied in the refinery model and are not necessary in proportion to physical measures (e.g., mass or energy content, etc.). In some cases, however, Azapagic and Clift (1999b) show that these LP-based allocation coefficients might coincide with some simple physical measures but it is completely different from arbitrarily choosing them in advance.

In relation (6), the product-related emissions α_i corresponds to the additional emissions of CO₂ attributable to the marginal production of the i th oil product b_i , *ceteris paribus*. To fully appreciate these marginal allocation coefficients let us focus on their mathematical formulation. To this end, we differentiate the emission balance constraint in Eq. (1) with respect to b_i as follows

$$\sum_{k \in \beta} E_k \left(\frac{dx_k}{db_i} \right) - \underbrace{\frac{d\varepsilon}{db_i}}_{=\alpha_i} = 0 \tag{7}$$

where the row-vector E_k corresponds to the input–emission coefficients as they appear in the basic index β and the vector (dx_k/db_i) represents the inverse of the marginal production of the k th unit process with respect to the i th oil product. As shown in Eq. (7), the α_i ($i \in M$) is calculated by

Table 1
Part of the optimal Simplex tableau of the refining example

↓	Basic variables	Slack gasoline	Slack diesel	Slack fuel oil	Slack emissions	Slack (x1)	Slack (x2)	Slack (x5)	Optimal value
	x_4	0.757	-6.111	7.373	0.000	-0.649	-1.911	0.000	17.130
	x_3	-5.492	12.638	-5.126	0.000	-0.961	-0.161	0.000	135.560
	x_2	0.000	0.000	0.000	0.000	0.000	1.000	0.000	15.000
	ε	0.174	-0.272	0.229	-1.000	0.014	-0.024	0.000	10.500
	x_5	5.871	-5.694	-1.186	0.000	0.637	1.105	1.000	51.490
	Slack(x_5)	-5.871	5.694	1.186	0.000	-0.637	-1.105	0.000	0.500
	x_1	0.000	0.000	0.000	0.000	1.000	0.000	0.000	45.000
	C^*	21.212	48.888	186.464	0.000	31.171	57.711	0.000	17,531.610

tracking emissions through individual active unit process x_k ($k \in \beta$), and distributing the resultant emissions to the i th oil product from that process without using any arbitrary measures. Moreover, the complex interactions and synergies among the active refining units and the oil products are captured by the marginal production inverse vector. As a conclusion, relation (6) as compared to the sub-process allocation approach, provides a more complete picture of the refining system including the substitution effects and interdependencies among inputs and outputs simultaneously.

Similarly, γ_j refers to the additional quantities of CO₂ attributable to an extra unit of the j th scarce capacity process \bar{x}_j , *ceteris paribus*. A more detailed discussion of these process-related coefficients is the subject of Section 3.

Before proceeding further, we illustrate these concepts within the following numerical example. Consider a very simplified oil refinery optimization model in which the refinery processes five different types of crude oils (x_1, x_2, \dots, x_5) to produce three main type of oil products: gasoline, diesel and fuel oil. Over a typical year, the refiner’s objective is to satisfy its output target for each of the oil products (100, 87 and 72 t) at minimum cost subject to the market crude prices (respectively, \$55, \$50, \$72, \$85 and \$60 per ton) and its prevailing linear technology. In the short-run, the availability of the crude oils 1, 2 and 5 is limited to 45, 15 and 52 t. We also assume that the quantity of CO₂ released by processing each ton of the crude oils are respectively 0.05, 0.04, 0.03, 0.06 and 0.05 t. Since the CO₂ allocation procedure does not depend upon the permit price, we do not introduce them into this simplified example. The cost minimization LP model of this refinery can be formulated as

$$\left\{ \begin{array}{l} \min 55x_1 + 50x_2 + 72x_3 + 85x_4 + 60x_5 \\ s.t. \\ 0.29x_1 + 0.19x_2 + 0.39x_3 + 0.35x_4 + 0.49x_5 \geq 100 \quad (\text{demand for gasoline}) \\ 0.34x_1 + 0.29x_2 + 0.35x_3 + 0.29x_4 + 0.29x_5 \geq 87 \quad (\text{demand for diesel}) \\ 0.34x_1 + 0.48x_2 + 0.25x_3 + 0.34x_4 + 0.19x_5 \geq 72 \quad (\text{demand for fuel oil}) \\ 0.05x_1 + 0.04x_2 + 0.03x_3 + 0.06x_4 + 0.05x_5 - \varepsilon = 0 \quad (\text{CO}_2 \text{ equation}) \\ x_1 \leq 45, \quad x_2 \leq 15, \quad x_5 \leq 52, \quad (\text{crudes availability}) \\ x_j \geq 0, \quad j = 1, \dots, 5, \end{array} \right.$$

where the technical coefficients $[a_{ij}]$ represent the average productivity of crudes in oil products.

Table 1 contains a part of the final Simplex tableau of the refining example. The column to the immediate left of the tableau indicates the basic activities, whose optimal values are read in the most right column. The first row corresponds to the slack variables associated with the model constraints. The last row represents the dual optimal variables (marginal production costs and

scarce crude opportunity costs). The optimal value of the cost (objective) function is located in the southeast corner of the final tableau. We recall that the following part of the optimal Simplex tableau corresponds to the basis inverse matrix B^{-1} .

At the optimum, all crude oils are processed, the marginal production cost of the oil products and the shadow prices of crude oils are respectively $y^T = [\$21.212, \$48.888, \$186.464]$ and $\lambda^T = [\$31.171, \$57.711, \$0.000]$ and the total accounting variable cost amounts to \$17,531.61. Lastly, the refinery discharges 10.5 t of CO₂. The vector e_ε^T is read as the bold row of the final tableau corresponding to the basic CO₂ variable ε :

$$e_\varepsilon^T B^{-1} = [0.174, -0.272, 0.229, -1.000, 0.014, -0.024, 0.000]$$

$\underbrace{\hspace{10em}}_{\alpha_i} \qquad \underbrace{\hspace{10em}}_{\gamma_j}$

and relation (5) is given by

$$\varepsilon = \underbrace{17.400}_{\text{gasoline}} - \underbrace{23.664}_{\text{diesel}} + \underbrace{16.488}_{\text{fuel oil}} + \underbrace{0.630}_{\text{crude 1}} - \underbrace{0.360}_{\text{crude 2}} = 10.50.$$

Within the retrospective WTT analysis, the refinery’s CO₂ emissions must be allocated solely among the final oil products. Consequently, whenever the non-product binding constraints are active (so $\gamma > 0$), the product-related allocations (i.e., $\alpha_i, i \in M$) underestimate or overestimate the total volume of the refinery’s CO₂ emissions. In an empirical LP study (Tehrani Nejad, submitted for publication), we observed that, in presence of active capacity constraints, the total allocated CO₂ to the refinery oil products based on their respective marginal contents were 3.5 times more than the total unallocated CO₂ emissions. Therefore, this handicap is a valid objection to LP as an allocation tool for retrospective WTT or WTW studies.

2.3. Reviewing the existing propositions

Before introducing our procedure in Section 3, we briefly review and discuss the existing propositions in the LP literature to this problem. For a Shapely value-based approach to this issue, see Pierru (in press-a).⁹

2.3.1. Azapagic–Clift proposition

To fully allocate the whole CO₂ emissions among output products,¹⁰ (Azapagic and Clift, 1999b) suggest optimising the LP model (1) without its process capacity and availability constraints. This proposition, however, considers a long-run production specification (provided that investment costs are included in the objective function, which is not the case in their proposition) and does not reflect the short-run environment of the refinery. It is important to note that, since the late 1990’s the European refining production configuration is not in accordance with the oil products’ demand market in Europe and is far from its long-run equilibrium situation. Since the allocation parameters should correspond to the real behavior of the refinery system,

⁹ The main practical problem associated with this approach is the need for efficient algorithms for computing the Shapely value.

¹⁰ In their papers, Azapagic and Clift use an emission minimization framework and, the CO₂ allocations correspond to the dual optimal variables associated with output products. However, we believe that the “emission efficient” allocations which are obtained from their approach do not correspond to the current “cost efficient” framework in which operate the refineries.

using a long-run specification in order to make short-run environmental policies is not reasonable and might lead to flawed conclusions (see Section 4 for a numerical example).

2.3.2. Ramsey–Boiteux pricing method

From the economic literature, Ramsey–Boiteux pricing (RBP) method (Ramsey 1927, Boiteux, 1956), used for public utility regulation, may seem to be another appealing suggestion to this issue (Pierru, in press-b). Let us recall that the basic idea behind RBP application is to derive the average product-related allocation coefficients in such a way that all final products be varied by the same proportion from the quantities that would be produced at prices equal to their corresponding marginal costs.

Nevertheless, the use of this method to deal with the carbon dioxide allocation issue can be criticized on three points. First, in most of the RBP applications, the substitution effects among output products are supposed to be zero. But in a refinery context, ignoring the substitution effects among oil products brings us back to the shortcoming of the sub-process approaches and might significantly skew the obtained CO₂ allocations. As discussed in the introduction, the higher pricing and taxing of gasoline has been a key determinant in dieselization trends of the light vehicle fleets and in the development of diesel vehicle technology in European countries. The diesel technology has advanced at an incredible pace in recent years, that European consumers now view diesel as the preferred alternative not only for its lower price but also for its superior performance (common rail technology, less consumption and diesel particulate filter). Therefore, the introduction of these substitution effects should not be regarded as “unnecessary complexity” but rather as informative signals from the consumers’ behavior pattern.

Second, this application involves numerical estimates of the consumer demand for final products (direct and cross elasticities). Fiertz and Monico (2004) illustrate the sensitivity of the RBP-based allocations to potential errors in the selection of exogenous elasticity values. Moreover, the unambiguity property of a non-arbitrary allocation rule might be also violated when different elasticities exist for the same oil product in the literature.

Third, the real connection between oil products b and process-related CO₂ emissions (i.e., $\sum_{j \in S} \gamma_j \bar{x}_j$) is based on technical relations embodied in the refinery structure and is completely independent from the oil products’ market structure. Therefore, reallocating the process-related emissions to oil products based on their respective price elasticities fall again in the realm of choosing arbitrarily a market value indicator in advance. In other words, RBP application implies that the CO₂ emissions released from the refinery unit processes vary in direct proportion to variation in the oil product prices. This assumption is too strong and simply wrong because the pollutant emissions inherent in the refinery system depend directly upon the technical sophistications of the production factors.

3. Reallocating the process-related CO₂ emissions among joint oil products

3.1. First step

For some technical reasons that will be discussed in Section 3.2, we begin the first step of our two-step methodology by introducing into model (1) a material balance constraint for the process losses. In the oil refining industry, due to various factors including evaporation and leakage, the quantity, weight or volume of oil products are less than that of crude oil inputs and feedstocks. In cost accounting systems, this is referred to as normal process loss and is usually expressed as a percentage of the input activity volume. In fact, this constraint already exists in most operational LP refinery models to capture the quantity of process losses.

Table 2
Part of the optimal Simplex tableau of the new refining example

↓	Basic variables	Slack gasoline	Slack diesel	Slack fuel oil	Slack losses	Slack emissions	Slack (x1)	Slack (x2)	Slack (x5)	Optimal value
	x_3	-5.492	12.638	-5.126	0.000	0.000	-0.961	-0.161	0.000	135.560
	x_1	0.000	0.000	0.000	0.000	0.000	1.000	0.000	0.000	45.000
	ε	0.174	-0.272	0.229	0.000	-1.000	0.014	-0.024	0.000	10.500
	x_5	5.871	-5.694	-1.186	0.000	0.000	0.637	1.105	1.000	51.490
	x_2	0.000	0.000	0.000	0.000	0.000	0.000	1.000	0.000	15.000
	Slack(x_5)	-5.871	5.694	1.186	0.000	0.000	-0.637	-1.105	0.000	0.500
	ℓ	0.136	-0.166	0.060	-1.000	0.000	0.026	0.033	0.000	5.190
	x_4	0.757	-6.111	7.373	0.000	-0.649	-0.649	-1.911	0.000	17.130
	C^*	21.212	48.888	186.464	0.000	0.000	31.171	57.711	0.000	17,531.610

Introducing the material balance constraint for losses, the new LP model takes on the following specifications

$$\begin{cases} \min & c^T x \\ \text{s.t.} & Ax \geq b \\ & l^T x - \ell = 0 \\ & E^T x - \varepsilon = 0 \\ & x \leq \bar{x} \\ & x \geq 0, \ell \geq 0, \varepsilon \geq 0. \end{cases} \tag{8}$$

where the n -vector $l^T = [l_1, l_2, \dots, l_n]$ represents the loss coefficients for each input activity, and the variable ℓ measures the total losses inherent in the production process. We also assume that there is no abnormal loss so that $e^T A_j + l_j = 1$ for all input activities, where $e^T = [1, 1, \dots, 1]$. The new refining example is:

$$\begin{cases} \min & 55x_1 + 50x_2 + 72x_3 + 85x_4 + 60x_5 \\ \text{s.t.} & 0.29x_1 + 0.19x_2 + 0.39x_3 + 0.35x_4 + 0.49x_5 \geq 100 \quad (\text{demand for gasoline}) \\ & 0.34x_1 + 0.29x_2 + 0.35x_3 + 0.29x_4 + 0.29x_5 \geq 87 \quad (\text{demand for diesel}) \\ & 0.34x_1 + 0.48x_2 + 0.25x_3 + 0.34x_4 + 0.19x_5 \geq 72 \quad (\text{demand for fuel oil}) \\ & 0.03x_1 + 0.04x_2 + 0.01x_3 + 0.02x_4 + 0.03x_5 - \ell = 0 \quad (\text{loss equation}) \\ & 0.05x_1 + 0.04x_2 + 0.03x_3 + 0.06x_4 + 0.05x_5 - \varepsilon = 0 \quad (\text{CO}_2 \text{ equation}) \\ & x_1 \leq 45, \quad x_2 \leq 15, \quad x_5 \leq 52, \quad (\text{crudes availability}) \\ & x_j \geq 0, \quad j = 1, \dots, 5. \end{cases}$$

The concept of the process-related emissions, $\gamma_j (j \in S)$, and their technical connection with the oil products are key to our analysis. For any $j \in S$, γ_j can be formulated by differentiating the emission balance constraint in Eq. (8) with respect to x_j as follows

$$\sum_{\substack{k \in \beta \\ k \neq \varepsilon}} E_k \left(\frac{dx_k}{dx_j} \right) - \underbrace{\frac{d\varepsilon}{dx_j}}_{=\gamma_j} = 0, \tag{9}$$

where the vector E_k corresponds to the emissions row in B , from which the (-1) coefficient associated with ε is omitted and the vector $(\frac{dx_k}{dx_j})$ corresponds to the column of B^{-1} associated with the slack variable of the j th scarce unit process, from which γ_j is extracted. In economic terms,

Table 3
Adjustment of oil products, emissions and losses to an extra unit of scarce crude 1

	MRTS	Gasoline	Diesel	Fuel oil	Emissions	Losses
<i>Increase</i>						
Crude 1	1.000	1.000×0.29	1.000×0.34	1.000×0.34	1.000×0.05	1.000×0.03
Crude 5	0.637	0.637×0.49	0.637×0.29	0.637×0.19	0.637×0.05	0.637×0.03
<i>Decrease</i>						
Crude 3	-0.961	-0.961×0.39	-0.961×0.35	-0.961×0.25	-0.961×0.03	-0.961×0.01
Crude 4	-0.649	-0.649×0.35	-0.649×0.29	-0.649×0.34	-0.649×0.06	-0.649×0.02
<i>Net effect</i>		0.000	0.000	0.000	0.014	0.026

$(\frac{dx_i}{dx_j})$ represents the vector of marginal rates of technical substitution (MRTS) between the j th scarce unit process and all the unit process activities involved in the production plan. More precisely, this vector shows the rate at which basic inputs should be substituted along the given oil product isoquant b , whenever an extra unit of j th scarce input were made available at the optimum. For notational convenience, we set $(\frac{dx_i}{dx_j}) = \zeta_j$. These marginal coefficients are a powerful tool to capture the technical characteristics between unit processes and oil product outputs at each stage of the refining process. To the best of our knowledge, and surprisingly, these coefficients have never been used in any empirical LP study.

In our numerical example, the vectors of MRTS for crude oils 1 and 2 are read as the bold column of the final Simplex tableau (Table 2) corresponding to slack variables slack (x_1) and slack (x_2):

$$\zeta_{x1} = [-0.961, 1.000, 0.637, 0.000, -0.637, 0.026, -0.649]^T,$$

$$\zeta_{x2} = [-0.161, 0.000, 1.105, 1.000, -1.105, 0.033, -1.911]^T.$$

The vectors ζ_{x1} and ζ_{x2} indicate respectively how the additional unit of scarce crude 1 and 2 should be best used, while maintaining oil product levels constant. In our example, an additional unit of scarce crude 1 would alter the optimal input requirement set (not the optimal technology B) by decreasing refining of crude oils 3 and 4 by 0.961 and 0.649 of tons, and increasing the use of crude 5 by 0.637 t. This substitution process would lead to 0.014 t of additional CO₂ emissions and is the most efficient use (from a cost point of view) of the extra unit of scarce crude 1. The same interpretation applies to ζ_{x2} . Now a deeper question is how these process-related emissions (0.014 and -0.024) should be reallocated among joint oil products?

Table 4
Adjustment of oil products, emissions and losses to an extra unit of scarce crude 2

	MRTS	Gasoline	Diesel	Fuel oil	Emissions	Losses
<i>Increase</i>						
Crude 1	1.000	1.000×0.19	1.000×0.29	1.000×0.48	1.000×0.04	1.000×0.04
Crude 5	1.105	1.105×0.49	1.105×0.29	1.105×0.19	1.105×0.05	1.105×0.03
<i>Decrease</i>						
Crude 3	-0.161	-0.161×0.39	-0.161×0.35	-0.161×0.25	-0.161×0.03	-0.161×0.01
Crude 4	-1.911	-1.911×0.35	-1.911×0.29	-1.911×0.34	-1.911×0.06	-1.911×0.02
<i>Net effect</i>		0.000	0.000	0.000	-0.024	0.033

To answer this question, let us first decompose, with the help of the MRTS vectors and the technical coefficients in model (8), the optimal adjustment of carbon dioxide emissions and process losses in respond to an additional unit of scarce crude oils 1 and 2 (see Tables 3 and 4).

Now to reveal the physical interactions between the marginal CO₂ content of *j*th scarce unit process and final oil products, we introduce the optimal adjustment of final outputs (oil products and losses) into relation (9), which leads to

$$\gamma_j = \sum_{\substack{k \in \beta \\ k \neq \varepsilon}} E_k \left(\sum_{i \in M} a_{ik} + l_k \right) \zeta_{jk} \tag{10}$$

where, by construction, $\sum_{i \in I} a_{ik} + l_k = 1$.

To separate the part of each output in γ_j , we introduce the following $(k-1) \times (k-1)$ allocating matrix Ω_i and Ω_ℓ :

$$\Omega_i = \text{diag}(a_{ik}, i \in M, k \in \beta, k \neq \varepsilon), \tag{11}$$

$$\Omega_\ell = \text{diag}(l_k, k \in \beta, k \neq \varepsilon), \tag{12}$$

where the coefficients a_{ik} are the average productivity of crude oils associated with the *i*th row of *B*. Similarly, the coefficients l_k correspond to the loss coefficients associated with the material balance row for losses in *B*.

In our numerical illustration, $\beta = \{x_3, x_1, \varepsilon, x_5, x_2, \text{slack}(x_5), \not\in x_4\}$ and, the optimal basis matrix *B* is given by

$$B = \begin{array}{cccccccc|l} 0.39 & 0.29 & 0.00 & 0.49 & 0.19 & 0.00 & 0.00 & 0.35 & \leftarrow & \text{Gasoline} \\ 0.35 & 0.34 & 0.00 & 0.29 & 0.29 & 0.00 & 0.00 & 0.29 & \leftarrow & \text{Diesel} \\ 0.25 & 0.34 & 0.00 & 0.19 & 0.48 & 0.00 & 0.00 & 0.34 & \leftarrow & \text{Fuel oil} \\ 0.01 & 0.03 & 0.00 & 0.03 & 0.04 & 0.00 & -1.00 & 0.02 & \leftarrow & \text{Loss} \\ 0.03 & 0.05 & -1.00 & 0.03 & 0.04 & 0.00 & 0.00 & 0.06 & \leftarrow & \text{Emissions} \\ 0.00 & 1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & \leftarrow & \text{Crude 1} \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 1.00 & 0.00 & \leftarrow & \text{Crude 2} \\ 0.00 & 0.00 & 0.00 & 1.00 & 0.00 & 1.00 & 0.00 & 0.00 & \leftarrow & \text{Crude 3} \end{array}$$

The allocating diagonal matrix for oil products and process loss are calculated as

$$\begin{aligned} \Omega_{\text{gasoline}} &= \text{diag}[0.39, 0.29, 0.49, 0.19, 0.00, 0.00, 0.00, 0.35], \\ \Omega_{\text{diesel}} &= \text{diag}[0.35, 0.34, 0.29, 0.29, 0.00, 0.00, 0.29], \\ \Omega_{\text{fuel oil}} &= \text{diag}[0.25, 0.34, 0.19, 0.48, 0.00, 0.00, 0.34], \\ \Omega_\ell &= \text{diag}[0.01, 0.03, 0.03, 0.04, 0.00, -1.00, 0.02]. \end{aligned}$$

Using definitions (11) and (12), relation (10) can be rewritten as

$$\gamma_j = E_B^T \left(\sum_{i \in M} \Omega_i + \Omega_\ell \right) \zeta_j \tag{13}$$

where by construction, $\sum_{i \in M} \Omega_i + \Omega_{\ell} = \text{diag}(1, 1, \dots, 1)$. Rearranging Eq. (13), we separate the part of oil product outputs from process loss in γ_j :

$$\gamma_j = \overbrace{\sum_{i \in M} E_B^T \Omega_i \zeta_j}^{\text{part of oil products}} + \overbrace{E_B^T \Omega_{\ell} \zeta_j}^{\text{part of loss}}. \tag{14}$$

Relation (14) relates each γ_j to the refinery’s outputs (including the process losses) through a realistic technical relationship which emerges from the equilibrium behavior of the firm.

As illustration, we calculate the part of each refining oil product and process losses in the marginal content of CO₂ emissions of crude 1:

$$\begin{aligned} \gamma_1 &= E_B^T(\Omega_{\text{gasoline}}) \zeta_{x1} + E_B^T(\Omega_{\text{diesel}}) \zeta_{x1} + E_B^T(\Omega_{\text{fuel oil}}) \zeta_{x1} + c_B^T(\Omega_{\ell}) \zeta_{x1} \\ \gamma_1 &= [0.03, 0.05, 0.05, 0.04, 0.0, 0.0, 0.06] \begin{bmatrix} 0.39 & & & & & & \\ & 0.29 & & & & & \\ & & 0.49 & & & & \\ & & & 0.19 & & & \\ & & & & 0 & & \\ & & & & & 0 & \\ & & & & & & 0.35 \end{bmatrix} \begin{bmatrix} -0.961 \\ 1.000 \\ 0.637 \\ 0.000 \\ -0.637 \\ 0.026 \\ -0.649 \end{bmatrix} \\ &+ [0.03, 0.05, 0.05, 0.04, 0.0, 0.0, 0.06] \begin{bmatrix} 0.35 & & & & & & \\ & 0.34 & & & & & \\ & & 0.29 & & & & \\ & & & 0.29 & & & \\ & & & & 0 & & \\ & & & & & 0 & \\ & & & & & & 0.29 \end{bmatrix} \begin{bmatrix} -0.961 \\ 1.000 \\ 0.637 \\ 0.000 \\ -0.637 \\ 0.026 \\ -0.649 \end{bmatrix} \\ &+ [0.03, 0.05, 0.05, 0.04, 0.0, 0.0, 0.06] \begin{bmatrix} 0.25 & & & & & & \\ & 0.34 & & & & & \\ & & 0.19 & & & & \\ & & & 0.48 & & & \\ & & & & 0 & & \\ & & & & & 0 & \\ & & & & & & 0.34 \end{bmatrix} \begin{bmatrix} -0.961 \\ 1.000 \\ 0.637 \\ 0.000 \\ -0.637 \\ 0.026 \\ -0.649 \end{bmatrix} \\ &+ [0.03, 0.05, 0.05, 0.04, 0.0, 0.0, 0.06] \begin{bmatrix} 0.01 & & & & & & \\ & 0.03 & & & & & \\ & & 0.03 & & & & \\ & & & 0.04 & & & \\ & & & & 0 & & \\ & & & & & -1 & \\ & & & & & & 0.02 \end{bmatrix} \begin{bmatrix} -0.961 \\ 1.000 \\ 0.637 \\ 0.000 \\ -0.637 \\ 0.026 \\ -0.649 \end{bmatrix} \\ \gamma_1 &= \underbrace{0.005233}_{\text{gasoline}} + \underbrace{0.004853}_{\text{diesel}} + \underbrace{0.002604}_{\text{fuel oil}} + \underbrace{0.001388}_{\text{process loss}} = 0.014. \end{aligned}$$

The same calculation applies to γ_2 :

$$\begin{aligned} \gamma_2 &= E_B^T(\Omega_{\text{gasoline}}) \zeta_{x2} + E_B^T(\Omega_{\text{diesel}}) \zeta_{x2} + E_B^T(\Omega_{\text{fuel oil}}) \zeta_{x2} + c_B^T(\Omega_{\ell}) \zeta_{x1} \\ \gamma_2 &= \underbrace{0.0073422}_{\text{gasoline}} - \underbrace{0.0073194}_{\text{diesel}} - \underbrace{0.0104944}_{\text{fuel oil}} + \underbrace{0.000916}_{\text{process loss}} = -0.024. \end{aligned}$$

We can verify that these two coefficients (0.014 and -0.024) appear in the final Simplex tableau (Table 2) under the heading of slack variables of crude 1 and 2 which are related to the CO_2 variable ε .

Now, using the decomposition relation (14), we rewrite the relation (6) as

$$\varepsilon = \sum_{i \in M} \alpha_i b_i + \sum_{j \in S} \left[\sum_{i \in M} E_B^T \Omega_i \varsigma_j + E_B^T \Omega_{\ell} \varsigma_j \right] \bar{x}_j$$

with some algebraic manipulations we get,

$$\varepsilon = \sum_{i \in M} \left[\alpha_i + \sum_{j \in S} E_B^T \Omega_i \varsigma_j \underbrace{\frac{\bar{x}_j}{b_i}}_{\delta_{ij}} \right] b_i + \sum_{j \in S} E_B^T \Omega_{\ell} \varsigma_j \underbrace{\frac{\bar{x}_j}{\ell}}_{\delta_{\ell j}} \ell. \tag{15}$$

The expressions $\Omega_i \varsigma_j$ and $\Omega_{\ell} \varsigma_j$ measure the total variation of the i th oil product and the process losses whenever an extra unit of j th scarce unit process is made available, *ceteris paribus*. Therefore, the expressions δ_{ij} and $\delta_{\ell j}$ correspond to the definition of the production and loss elasticities of the j th scarce unit process at the optimum. Simplifying relation (15), we get

$$\varepsilon = \sum_{i \in M} \left(\overbrace{\alpha_i + \sum_{j \in S} E_B^T \delta_{ij}}^{\bar{\alpha}_i} \right) b_i + \left(\overbrace{\sum_{j \in S} E_B^T \delta_{\ell j}}^{\bar{\theta}} \right) \ell. \tag{16}$$

The economic interpretation of relation (16) runs as follows: at the optimal solution of model (8), the carbon dioxide emissions of the refinery is quasi fully allocated among the oil products through their marginal contributions α_i and the production elasticity δ_{ij} of the scarce unit processes. The obtained (re)allocation coefficients $\bar{\alpha}_i$ depend totally upon the technical and physical relationships that define the operating state of the refinery and accounts for all interdependencies in the production plan. Contrary to the RBP formula, the numerical values of different elasticities are directly provided by the final Simplex tableau.

From relation (16), we observe that a rather small residual part remains still unallocated to the oil products, i.e., $\bar{\theta} \ell$. We define the relative error term of the first step by $\bar{\theta} \ell / \varepsilon$, which is essentially proportionate to the normal loss coefficients. While in almost real-world refinery production models the loss coefficients are small, the relative error of this first step should be very small too. So in empirical studies, $\bar{\alpha}_i$ is a good estimator of the average contribution of the i th oil product to the refinery’s CO_2 emissions and the reallocation procedure can be stopped here. Referring back to the refinery example, $\bar{\alpha}$ and $\bar{\theta}$ are calculated as follows

$$\bar{\alpha} = \begin{bmatrix} 0.1742 \\ -0.2722 \\ 0.2293 \end{bmatrix} + \begin{bmatrix} 0.001253880 \\ 0.001248410 \\ -0.000558580 \end{bmatrix} = \begin{bmatrix} 0.17545 \\ -0.27090 \\ 0.22874 \end{bmatrix}$$

It can be easily verified that the obtained vectors $\bar{\alpha}$ and $\bar{\theta}$ fully partition the CO₂ emissions of the refinery (10.50 t) among final oil products and the process losses. By rounding off to the nearest hundred,

$$\varepsilon = \sum_{i \in M} \bar{\alpha}_i b_i + \bar{\theta} \ell = 10.44 + 0.06 = 10.50$$

Here, the relative error term of the first step is 0.72%.

3.2. Second step

As shown in Eq. (16), the reallocation scheme developed in the first step is quasi additive due to the presence of the refinery’s process losses. An additive CO₂ allocation pattern requires reassigning the loss-related emissions $\bar{\theta} \ell$ over final products. The problem considered in the second step is how much of $\bar{\theta} \ell$ should be shared by each product.

The quantity of the refinery loss ℓ is a basic variable (because always positive) and can be allocated among the primal constraints in the same way as we did for the emissions variable ε . In fact, this kind of analysis is always applicable whenever the objective of the study is concerned with allocating the optimal value of a basic variable over a sub-set of the RHS. By an analogous logic to that developed in the first step, we decompose the loss variable ℓ of the LP model (8) according to the optimal basic variable definition:

$$\ell = e_{\ell}^T B^{-1} \begin{bmatrix} b \\ 0 \\ 0 \\ \bar{x} \end{bmatrix}, \tag{17}$$

where e_{ℓ} is the ℓ th unit vector ($e_{\ell t} = 0$ for $t \neq \ell$ and $e_{\ell \ell} = 1$) and $e_{\ell}^T B^{-1}$ corresponds to the row of B^{-1} associated with the basic variable ℓ and contains several blocks referred to the slack variables of model (8). Rewriting Eq. (17), we get the following relation

$$\ell = \sum_{i \in M} \vartheta_i b_i + \sum_{j \in S} v_j \bar{x}_j, \tag{18}$$

that relates the total process loss of the refinery to the oil products b and the limited (fixed) unit processes \bar{x} through the marginal allocation coefficients, ϑ_i ($i \in M$) and v_j ($j \in S$). In words, ϑ_i corresponds to the marginal loss content of i th final product, referring to the additional quantities of loss due to its marginal production. Similarly, v_j refers to the additional quantities of loss attributable to an extra unit of the j th scarce unit process. According to the optimal technology, ϑ_i and v_j can be negative, zero or positive.

Before proceeding further, we illustrate these concepts within our numerical example. The marginal coefficients ϑ_i and v_j are read as the row of the final Simplex tableau (see Table 2) corresponding to the basic variable ℓ :

$$e_{\ell}^T B^{-1} = \left[\underbrace{0.136, -0.166, 0.060}_{\vartheta_i (i \in M)} \quad -1.00, 0.000, \underbrace{0.026, 0.033, 0.000}_{v_j (j \in S)} \right],$$

and relation (17) is given by

$$\ell = \underbrace{13.600}_{\text{gasoline}} - \underbrace{14.442}_{\text{diesel}} + \underbrace{4.320}_{\text{fuel oil}} + \underbrace{1.170}_{\text{crude 1}} + \underbrace{0.495}_{\text{crude 2}} = 5.19.$$

Relating the refinery loss to solely oil products needs reallocating the loss contribution of scarce unit process over oil products. By analogy to γ_j , the marginal coefficient v_j can be formulated by differentiating the loss constraint in Eq. (8) with respect to x_j as follows

$$\sum_{\substack{k \in \beta \\ k \neq \ell}} l_k \left(\frac{dx_k}{dx_j} \right) - \underbrace{\frac{d\ell}{dx_j}}_{=v_j} = 0, \tag{19}$$

where the vector l_k corresponds to the loss row in B , from which the (-1) coefficient associated with l is omitted. The vector $\left(\frac{dx_k}{dx_j}\right)$ corresponds to the column of B^{-1} associated with the slack variable of the j th scarce unit process, from which v_j is extracted. For notational convenience, we set $\left(\frac{dx_k}{dx_j}\right) = \tilde{\zeta}_j$.

For illustration purposes, the relation (19) can be verified as follows

$$v_{x1} = [0.01, 0.03, 0.00, 0.03, 0.04, 0.00, 0.02] \begin{bmatrix} -0.961 \\ 1.000 \\ 0.014 \\ 0.637 \\ 0.000 \\ -0.637 \\ -0.649 \\ -0.161 \\ 0.000 \\ -0.024 \\ 1.105 \\ 1.000 \\ -1.105 \\ -1.911 \end{bmatrix} = 0.026,$$

$$v_{x2} = [0.01, 0.03, 0.00, 0.03, 0.04, 0.00, 0.02] \begin{bmatrix} -0.161 \\ 0.000 \\ -0.024 \\ 1.105 \\ 1.000 \\ -1.105 \\ -1.911 \end{bmatrix} = 0.033.$$

We can verify that these two coefficients (0.026 and 0.033) appear in the final Simplex tableau (Table 2) under the heading of slack variables of crudes 1 and 2 which are related to the basic loss variable ℓ .

Again by an analogy to the first stage, separating the part of each oil product in v_j requires to introduce the following $(k-1) \times (k-1)$ allocating matrix $\tilde{\Omega}_i$

$$\tilde{\Omega}_i = \text{diag}(a_{ik}, i \in M, k \in \beta, k \neq \ell). \tag{20}$$

In our refining model, the new oil products allocating matrix are given by

$$\begin{aligned} \tilde{\Omega}_{\text{gasoline}} &= \text{diag}[0.39, 0.29, 0.00, 0.49, 0.19, 0.00, 0.35], \\ \tilde{\Omega}_{\text{diesel}} &= \text{diag}[0.35, 0.34, 0.00, 0.29, 0.29, 0.00, 0.29], \\ \tilde{\Omega}_{\text{fuel oil}} &= \text{diag}[0.25, 0.34, 0.00, 0.19, 0.48, 0.00, 0.34]. \end{aligned}$$

Using definition (20), relation (19) can be rewritten as

$$v_j \approx l_B^T \left(\sum_{i \in M} \tilde{\Omega}_i \right) \tilde{\zeta}_j, \tag{21}$$

where $\sum_{i \in M} \tilde{\Omega}_i \approx \text{diag}(1, 1, \dots, 1)$.

Using relation (22), we rewrite the loss respond function (18) as

$$\ell = \sum_{i \in M} \left[\vartheta_i + \sum_{j \in S} l_B^T \underbrace{\tilde{\Omega}_i^n \tilde{\zeta}_j \frac{\bar{x}_j}{b_i}}_{\delta_{ij}^n} \right] b_i. \tag{23}$$

The expression $\tilde{\Omega}_i^n \tilde{\zeta}_j$ measures the total variation of the i th oil product (with the normalized productivity coefficients) whenever an extra unit of j th scarce unit process is made available, *ceteris paribus*. Therefore, the expression δ_{ij}^n corresponds to the production elasticity of the j th scarce unit process at the optimum.

Simplifying Eq. (23), we obtain the following relation,

$$\ell = \sum_{i \in M} \left(\overbrace{\vartheta_i + \sum_{j \in S} l_B^T \delta_{ij}^n}^{\bar{\vartheta}_i} \right) b_i \tag{24}$$

whose economic interpretation runs as follows: at the optimal solution of the model (8), the total process loss of the refinery is fully allocated among the oil products through their marginal contributions ϑ_i and the production elasticity of scarce unit processes δ_{ij}^n .

In the numerical example,

$$\bar{\vartheta} = \begin{bmatrix} 0.1364 \\ -0.1667 \\ 0.0606 \end{bmatrix} + \begin{bmatrix} 0.006145 \\ 0.006392 \\ -0.007249 \end{bmatrix} = \begin{bmatrix} 0.142545 \\ -0.160308 \\ 0.067849 \end{bmatrix}.$$

It can be easily verified that the computed vector $\bar{\vartheta}$ fully partitions the total loss of the refinery among final oil products:

$$\ell = \bar{\vartheta}_i b_i = \underbrace{14.2545}_{\text{gasoline}} - \underbrace{13.9469}_{\text{diesel}} + \underbrace{4.8851}_{\text{fuel oil}} = 5.1927.$$

At this stage, we are finally ready to extract the exact average contribution of each oil product to the refinery’s carbon dioxide emissions. By using the relation (24) in Eq. (16) and doing some simple algebraic manipulations, we get

$$\varepsilon = \sum_{i \in M} \underbrace{\left(\alpha_i + \sum_{j \in S} E_B^T \delta_{ij} + \bar{\theta} \bar{\vartheta}_i \right)}_{\alpha_i^A} b_i \tag{25}$$

where the expression $\sum_{j \in S} E_B^T \delta_{ij} + \bar{\theta} \bar{\vartheta}_i$ represents the net contribution of the i th oil product to the process-related carbon dioxide emissions (i.e., $\sum_{j \in S} \gamma_j x_j$ in relation (6)). These net contributions are based on the production elasticity of the unit processes involved in the production plan and vary following the optimal technology of the multi-product refinery.

Let us re-state the meaning of relation (25) as follows: in a competitive situation and within a linear production technology, the whole CO₂ emissions of the refinery can be fully assigned to the

Table 5
Optimal Simplex tableau: long-run model

↓	Basic variables	x_2	x_3	x_4	Slack gasoline	Slack diesel	Slack fuel oil	Optimal value
	Slack (fuel oil)	-0.166	0.062	-0.092	-0.412	1.351	1.000	4.369
	x_1	1.054	0.708	0.492	-3.515	5.939	0.000	165.212
	ε	0.001	0.024	-0.014	0.030	0.121	0.000	13.576
	ℓ	-0.015	0.022	0.007	0.018	0.073	0.000	8.145
	x_5	-0.236	0.377	0.423	4.121	-3.515	0.000	106.303
	C^*	6.182	10.448	32.551	53.939	115.757	0.000	15,464.850

oil products through the “average allocation” coefficients α_i^A ($i \in M$). These latter include the *direct* and the *indirect* contribution of each oil product to the refinery’s CO₂ emissions. The direct contribution α_i corresponds to the marginal CO₂ content of the *i*th oil product and is directly obtainable from the final Simplex tableau as explained in Section 2. The indirect contribution, $\sum_{j \in S} E_B^T \delta_{ij} + \bar{\theta} \bar{\nu}_i$, depends upon the production elasticity of unit processes and should be calculated, *ex-post*, at the optimal solution of the LP model. Commercially available software for large scale models include some especial commands to extract these marginal coefficients from the optimal Simplex tableau.¹¹ Note that both the direct α_i and indirect $\sum_{j \in S} E_B^T \delta_{ij} + \bar{\theta} \bar{\nu}_i$ contributions are based on the same cost-efficient equilibrium (i.e., the same final simplex tableau) and are perfectly coherent with each other.

In the refining example, the average contribution of oil products to the CO₂ emissions of the refinery are given by

$$\alpha^A = \begin{bmatrix} 0.17545 \\ -0.27090 \\ 0.22874 \end{bmatrix} + (0.007259) \begin{bmatrix} 0.142545 \\ -0.160310 \\ 0.067849 \end{bmatrix} = \begin{bmatrix} 0.176 \\ -0.271 \\ 0.229 \end{bmatrix},$$

and the relation (25) is easily verified by

$$\varepsilon = \sum_{i \in M} \alpha_i^A b_i = \underbrace{17.600}_{\text{gasoline}} - \underbrace{23.577}_{\text{diesel}} + \underbrace{16.488}_{\text{fuel oil}} = 10.50.$$

In a large scale LP model, Tehrani Nejad M. and Saint-Antonin (2007) apply this two-stage procedure to the IFP (*Institut Français du Pétrole*) typical oil refinery.¹² The objective of the study is to assess the impact of reducing sulphur in European automotive fuels on the CO₂ content associated with gasoline and diesel. We show that, contrary to the traditional accounting allocation methods, gasoline has not always a higher average CO₂ content than that of diesel within the refinery.

4. Short-run vs. long-run CO₂ allocation coefficients

As described in Section 2.3, Azapagic and Clift (1999b) suggest optimising the LP model (1) without its process capacity and availability constraints in order to fully allocate the whole CO₂ emissions among oil products. Although their proposition is much easier to implement, it belongs to

¹¹ For instance, the available command in LAMPS (Linear And Mathematical Programming System) is TRANSFORMCOLUMNS (Advanced Mathematical Software Ltd., 1991).

¹² It corresponds to a typical European fluid catalytic cracking refinery of 10 million of tons with 700 constraints and more than 1500 variables.

Table 6
Short-run vs. long-run CO₂ allocations

Oil products	Short-run		Long-run	
	Marginal costs (\$/ton)	CO ₂ allocations (tonCO ₂ /ton)	Marginal costs (\$/ton)	CO ₂ allocations (tonCO ₂ /ton)
Gasoline	21.21	0.174	53.93	0.030
Diesel	48.88	-0.272	115.75	0.121
Fuel oil	186.46	0.229	0.00	0.000
CO ₂ emissions	10.50		13.57	

a long-run production specification and does not reflect the short-run environment of the refinery. To show that their procedure might lead to flawed results, we compute the long-run LP-based allocations of our refinery example by omitting the availability constraints of crudes.

Table 5 contains a part of the final Simplex tableau associated with the long-run refining example.

As shown in Table 5, the optimal production plan (optimal primal variables) and the optimal technology (the basis) adopted for a long-run production differs completely from the ones adopted for a short-run production plan (compare Table 2 with Table 5). In the long-run specification, the refiner processes only crudes 1 and 5 (due to their relative lower costs) in a quantity exceeding their availability in the short-run (165.212 and 106.303 t). The reduced cost of processing the non basic crude oils 2, 3 and 4 are respectively \$6.181, \$10.448 and \$32.551. This long-run production plan leads to an over production of 4.369 t and consequently to a zero marginal cost associated with the fuel oil. Furthermore, since the emission contents associated with crudes 1 and 5 are relatively higher, the refinery discharges 3.07 t more CO₂ emissions in the long-run.

In Table 6, we summarize the marginal cost and marginal allocation structures associated with the three oil products. As discussed before, the interesting feature of the LP-based allocations is the fact that they depend on the operating state of the refinery and reflect the underlying technical difficulties embodied in the model. From the marginal cost and allocation structures in Table 6, we notice that the refiner has much greater difficulties in meeting the fuel oil demand in the short-run as compared to the long-run. The overproduction of fuel oil in the long-run (which is exported in real LP refinery models) leads to a zero CO₂ allocation due to the fact that its additional demand would be simply met by decreasing the inventory (or exportation) level of this product (i.e., its basic slack variable) without any extra refining activity and extra CO₂ emissions. The same conclusion is true when a given oil product is imported at the optimum.

Moreover, the strategy to meet an additional demand of diesel also differs in the two specifications. Based on the short-run technology (in Table 2), the marginal production of diesel would alter the optimal crude requirement set by replacing the crudes 4 and 5 (which have the highest CO₂ contents) with crude 3 which is the less polluting one. This optimal substitution leads to a negative CO₂ content for diesel in the short-run. But the additional demand of diesel is met by adopting different techniques in the long-run leading to an increase of CO₂ emissions of 0.121 t. The same analysis applies to gasoline. This simple example illustrates that, contrary to what is suggested by Azapagic and Clift, and probably others, using a long-run specification for a short-run allocation issue can result in misleading policies.

5. Further remark

As explained in Section 4, in LP models, whenever a finished product is imported or exported at the optimum, its associated CO₂ content α_i would be systematically equal to zero. Indeed, this

is not a satisfactory result for determining tax rates on imported products based on their marginal environmental damage as noted by Babusiaux (2003).

In relation (25), since the expression $\sum_{j \in S} E_B^T \delta_{ij} + \bar{\theta} \bar{\vartheta}_i$ is always different to zero (because of the mathematical definition of non basic variables), the obtained average allocation coefficients α_i^A would be also different to zero even in cases where an oil product is imported or exported.

6. Conclusion

In this paper, we tackled a well-known polemical problem of how to allocate the CO₂ emissions of a petroleum refinery to its petroleum joint products. In effect, due to the complex interdependency, synergy and interaction effects among the various inputs and outputs, it is very difficult to establish any noncontroversial CO₂ allocation scheme based on the traditional approaches used so far in retrospective WTT and WTW studies.

The main objective of this paper was to show that how the information associated with an optimal LP solution can be properly used in order to fully allocate the CO₂ emissions of a refinery to its joint products, without having to need any further assumptions or information. To this end, we provided an original two-stage procedure, based on the marginal coefficients of the final Simplex tableau, to extract the average allocation coefficients associated with various oil products. These average coefficients include the direct and indirect contribution of each oil product to the refinery's CO₂ emissions; and, they depend totally upon the technical and physical relationships that define the operating state of the refinery. Therefore, they are perfectly consistent with the ISO 14041 recommendations and relevant for WTT purposes; especially when the goal of the study is either an efficiency judgment or prescribing policy changes.

To the best of our knowledge, these marginal coefficients have never been used in any LP-based allocation methods. A step-by-step numerical example was provided to illustrate the procedure.

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Appendix A

The ISO 14041 recommendations can be summarized as follows (Frischknecht, 2000)

- (1) Whenever possible allocation should be avoided;
- (2) Where allocation cannot be avoided, the system inputs and outputs should be partitioned between its different products or functions in a way which reflects the underlying physical relationships between them;
- (3) Where physical relationships alone cannot be established or used as the basis for allocation the inputs should be allocated between the outputs and functions in a way which reflects other relationships between them.

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